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Uniqueness of the Gaussian Quadrature for a Ball

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We construct a formula for numerical integration of functions over the unit ball in $\mathbb{R}^d$ that uses n Radon projections of these functions and is exact for all algebraic polynomials in $\mathbb{R}^d$ of degree $2n-1$. This is the highest algebraic degree of precision that could be achieved by an n term integration rule of this kind. We prove the uniqueness of this quadrature. In particular, we present a quadrature formula for a disk that is based on line integrals over n chords and integrates exactly all bivariate polynomials of degree $2n-1$. © 2000 Academic Press

Key Words: Gauss quadrature formula; orthogonal polynomials; highest degree of precision.

1. INTRODUCTION

There is a huge number of papers dealing with numerical integration of multivariate functions. In particular, explicit quadrature formulae have been produced for integration over simple domains Ω in the d-dimensional Euclidian space $\mathbb{R}^d$ like a ball, sphere, cube, or simplex (see, for example, [7, 11]). The integration rules are usually based on the evaluation of a finite

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number of linear functionals $L_1, ..., L_n$ of a given class, as point values, integrals over hyperplanes or spheres. One of the central trends in numerical integration is dealing with the construction and characterization of quadrature formulae of preassigned type

$$\int_\Omega f \approx \sum_{k=1}^{n} c_k L_k(f) \tag{1.1}$$

which have maximal *algebraic degree of precision* (abbreviated to ADP), that is, which are exact for all algebraic polynomials in d variables of degree as high as possible. The problem originates from Gauss [4] and his remarkable quadrature formula

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{k=1}^{n} A_k f(\tau_k),$$

which is exact for all univariate polynomials of degree $2n-1$. The nodes $\{\tau_k\}_1^n$ are situated at the zeros of the nth Legendre polynomial.

Formulae of the given form (1.1) are called *Gaussian* (or of *Gaussian type*) if they have a maximal ADP with respect to the corresponding polynomial space in $\mathbb{R}^d$.

The extension of the Gauss' result to the multivariate case encounters serious difficulties. Even in the simplest multivariate case, that of integration over plane domains Ω, there are only a few results which give in a closed form quadrature formulae of Gaussian type. Recently Xu in [13] (see also [2]) showed that a certain formula for integration over the cube in $\mathbb{R}^2$, based on point evaluations at a grid produced by the extremal points of Tchebycheff polynomials of first kind T_n, is minimal. However, the question of uniqueness of this quadrature formula is still open.

Another example, attributed to Lusternik and Kantorovich, comes from Mysovskih's book [7] (see [3] for multiple node extensions of this formula). It concerns integration over the disc D using integrals over n circles centered at the origin. The quadrature is exact for all polynomials from $\pi_{4n-1}(\mathbb{R}^2)$. It can be easily seen that there is no other quadrature based on n circles, co-centered at the origin, that has the same or higher ADP. But the question of uniqueness of the Gaussian quadrature of this type which uses any n circles contained in D is still open. Briefly, there is no result in the theory of multivariate numerical integration that gives in explicit form a Gaussian formula based on n pieces of information of a preassigned type and completely characterizes the constructed formula. The problem of uniqueness of the extremal formula is the most difficult part in such a characterization. Usually the uniqueness problem is reduced to the study of multivariate polynomials obeying orthogonal properties of a specific kind.

The aim of this paper is to give such an example of a multivariate Gaussian formula.

We extend the results from [1], where we studied quadrature formulae for the unit disc

$$D := \{(x, y) : x^2 + y^2 \leqslant 1\},$$

which are based on integrals over n chords. In the present paper we prove that there is a unique (up to rotation) quadrature formula of this kind which has a maximal ADP with respect to the space of bivariate polynomials. Moreover, we characterize completely the extremal chords by the zeros of a certain orthogonal polynomial. In this sense, the constructed formula can be viewed as a bivariate analogue of the Gauss formula.

Let us describe the result more precisely. Given the parameters (t_k, θ_k), $k = 1, ..., n$, we define the chords (see Fig. 1)

$$I_k := I(t_k, \theta_k) = \{(x, y) : x \cos \theta_k + y \sin \theta_k = t_k\} \cap D, \qquad k = 1, ..., n.$$

The corresponding linear polynomial, associated with I_k, will be denoted by l_k, namely

$$l_k = l_k(x, y) := x \cos \theta_k + y \sin \theta_k - t_k.$$

Everywhere in this paper the parameters θ_k are supposed to satisfy the requirement $\theta_k \in [0, \pi)$. We study quadrature formulae of the form

$$\iint_D f(x, y)\, dx\, dy \approx \sum_{k=1}^{n} A_k \int_{I_k} f(x, y)\, ds \tag{1.2}$$

which assign an approximate value to the double integral over D using a given finite number of line integrals

$$\int_{I_k} f := \int_{I_k} f(x, y)\, ds$$
$$:= \int_{-\sqrt{1-t_k^2}}^{\sqrt{1-t_k^2}} f(t_k \cos \theta_k - s \sin \theta_k,\ t_k \sin \theta_k + s \cos \theta_k)\, ds.$$

We consider the extremal problem of determining those coefficients $\{A_k\}_1^n$ and node chords $\{I_k\}_1^n$, for which the corresponding quadrature (1.2) integrates exactly all polynomials in two variables of degree as high as possible. An application of (1.2) to the associated polynomial ω^2, $\omega := l_1 \cdots l_n$, shows that its highest degree of precision is at most $2n-1$.

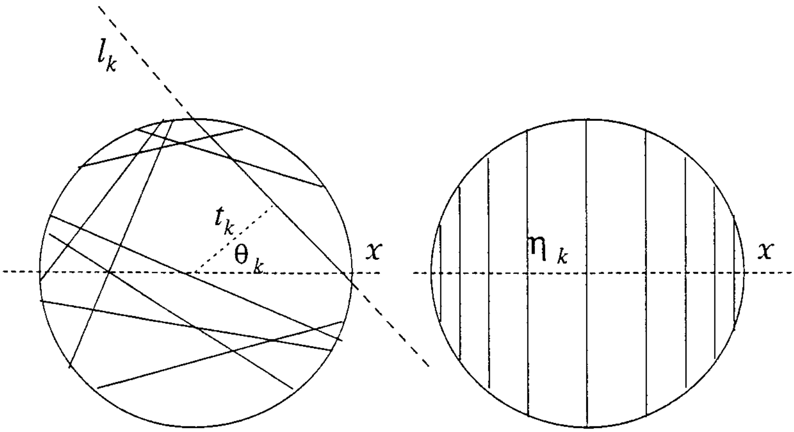

FIGURE 1

In [1] we proved the following. Let U_n be the Tchebycheff polynomial of second kind of degree n, that is,

$$U_n(\cos\theta) := \frac{\sin(n+1)\,\theta}{(n+1)\sin\theta}.$$

Let $\eta_1, ..., \eta_n$ be the zeros of U_n; $\eta_k = \cos(k\pi/(n+1))$, $k = 1, ..., n$.

THEOREM A. *The quadrature formula*

$$\iint_D f(x, y)\, dx\, dy \approx \sum_{k=1}^{n} A_k \int_{-\sqrt{1-\eta_k^2}}^{\sqrt{1-\eta_k^2}} f(\eta_k, y)\, dy, \tag{1.3}$$

with

$$A_k = \frac{\pi}{n+1} \sin\frac{k\pi}{n+1}, \qquad k = 1, ..., n,$$

is exact for each polynomial $f \in \pi_{2n-1}(\mathbb{R}^2)$.

In other words, among all the variety of quadrature formulae that use n chords the quadrature (1.3), based on the chords that are parallel to Oy and pass through the zeros of the Tchebycheff polynomial of second kind U_n, has a highest degree of precision.

The natural question then arises: Is (1.3) the only one (up to rotation) with this extremal property? We gave in [1] various characterization properties of the extremal set of chords and the associated polynomial ω. For example, a simple consequence of the extremality of (1.3) is that ω must be orthogonal on D to every polynomial from $\pi_{n-1}(\mathbb{R}^2)$. But as is known, there are many polynomials of degree n that are orthogonal to $\pi_{n-1}(\mathbb{R}^2)$.

In this paper, based on the results from [1], we prove the uniqueness of the Gaussian quadrature formula (1.3). Following the same idea, we construct and give a complete characterization of the Gauss–Lobatto type quadrature formulae for the disc D that use n chords and 2 points on the circumference ∂D.

We give also a general multivariate analogue of (1.3) for integration over the unit ball $\mathbf{B}^d$ in $\mathbb{R}^d$ using integrals over the intersection of $\mathbf{B}^d$ with n hyperplanes. This is exactly the type of information used in series of applications, for example, in the computer tomography performed for the purposes of medical research.

2. UNIQUENESS

We find it worthy to sketch first the idea of the proof. Observe that the number n of evaluations is much smaller than the dimension of $\pi_{2n-1}(\mathbb{R}^2)$, which is $n(2n+1)$. Thus, one may hope to find sufficient number of linearly independent polynomials in this wide class of exactness for which the data $\int_{I_k} f$ does not depend on the parameter θ_k. Luckily, the radial polynomials $(x^2+y^2)^m$, $m=0, ..., n-1$, have this property. Then, for radial polynomials, the information vector $(\int_{I_1} f, ..., \int_{I_n} f)$ does not depend on the angles $\theta_1, ..., \theta_n$ and consequently we may choose $\theta_1 = \cdots = \theta_n = 0$. This reduces the general case to the case of parallel chords and the multidimensional formula is reduced to a univariate one, which is exact for the even polynomials t^{2m}, $m=0, ..., n-1$. Next we use the symmetry of any extremal formula to show that the resulted univariate quadrature is exact also for the odd polynomials and thus for all polynomials of degree $2n-1$. Then it should be Gaussian and hence determined uniquely.

The symmetry of the node chords was established in [1]. This is an important point for our method. That is why we cannot apply it to non-symmetric formulae like those of Gauss–Radau type, although a quadrature of Gauss–Radau type (that uses n line integrals and a function value at a point from ∂D) can be easily constructed.

Our central result is the following theorem.

THEOREM 2.1. *There is a unique* (*up to rotation*) *quadrature formula of the form*

$$\iint_D f \approx \sum_{k=1}^{n} A_k \int_{I_k} f \tag{2.1}$$

which is exact for all polynomials from $\pi_{2n-1}(\mathbb{R}^2)$.

Proof. Assume that (2.1) is Gaussian. Then it integrates exactly the polynomials

$$(x^2+y^2)^m, \qquad m=0, ..., n-1.$$

We calculate

$$\iint_D (x^2+y^2)^m = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^m \, dy \, dx$$
$$= \int_{-1}^{1} \sqrt{1-x^2} \, p_{2m}(x) \, dx,$$

where p_{2m} is an even algebraic polynomial of degree exactly $2m$. On the other hand we derive that

$$\int_{I_k} (x^2+y^2)^m = \int_{-\sqrt{1-t_k^2}}^{\sqrt{1-t_k^2}} (t_k^2+y^2)^m \, dy = \sqrt{1-t_k^2} \, p_{2m}(t_k),$$

because $(x^2+y^2)^m$ is a radial polynomial and hence the line integral over I_k is the same as the integral over the line segment in D with equation $x=t_k$.

Therefore we have

$$\int_{-1}^{1} \sqrt{1-x^2} \, p_{2m}(x) \, dx = \sum_{k=1}^{n} A_k \sqrt{1-t_k^2} \, p_{2m}(t_k), \qquad m=0, ..., n-1.$$

Then the quadrature formula

$$\int_{-1}^{1} \sqrt{1-x^2} \, f(x) \, dx \approx \sum_{k=1}^{n} a_k f(t_k) \qquad \left(\text{with } a_k = A_k \sqrt{1-t_k^2}\right) \tag{2.2}$$

integrates exactly all even polynomials f of degree $\leqslant 2n-2$.

Now recall that the orthogonal polynomial $\omega := l_1 \cdots l_n$ associated with formula (2.1) has the property $\omega(-x, -y) = (-1)^n \, \omega(x, y)$ (see [1], Lemma 8). This implies that the zero lines of ω are pair-wise symmetric and consequently

$$t_k = -t_{n-k+1}, \qquad k=1, ..., n.$$

Let us denote by $J \subset \{1, 2, ..., n\}$ the set of indices k for which $t_k = 0$. Note that by Lemma 8 in [1], $J \neq \{1, 2, ..., n\}$. We apply the Gaussian quadrature formula (2.1) to the polynomials $\omega_k^2 := (l_1 \cdots l_{k-1} l_{k+1} \cdots l_n)^2$ and ω_{n-k+1}^2, for $k \notin J$, and get

$$\iint_D \omega_k^2 = A_k \int_{I_k} \omega_k^2 \quad \text{and} \quad \iint_D \omega_{n-k+1}^2 = A_{n-k+1} \int_{I_{n-k+1}} \omega_{n-k+1}^2.$$

We have that

$$\iint_D \omega_k^2 - \iint_D \omega_{n-k+1}^2$$
$$= -4t_k \iint_D \omega_{k,\,n-k+1}^2(x, y)(x \cos \theta_k + y \sin \theta_k)\, dx\, dy, \qquad k \notin J,$$

where $\omega_{k,\,n-k+1}^2 := (l_1 \cdots l_{k-1} l_{k+1} \cdots l_{n-k} l_{n-k+2} \cdots l_n)^2$ is a symmetric polynomial, that is, $\omega_{k,\,n=k+1}^2(-x, -y) = \omega_{k,\,n-k+1}^2(x, y)$. But

$$\iint_D xq(x, y)\, dx\, dy = \iint_D yq(x, y)\, dx\, dy = 0 \tag{2.3}$$

for every polynomial q with the property $q(-x, -y) = q(x, y)$, and therefore

$$\iint_D \omega_k^2 = \iint_D \omega_{n-k+1}^2.$$

On the other hand, again from the symmetry of $\omega_{k,\,n-k+1}^2$ and the lines I_k and I_{n-k+1}, we derive

$$\int_{I_k} \omega_k^2 = \int_{I_{n-k+1}} \omega_{n-k+1}^2,$$

and hence $A_k = A_{n-k+1}$, $k \notin J$. The latter is equivalent to

$$a_k = a_{n-k+1}, \qquad k \in \{1, ..., n\} \backslash J.$$

Thus the coefficients of the quadrature formula (2.2) that correspond to non-zero nodes are symmetric. Then (2.2) should be exact for all odd functions, and particularly for all odd polynomials of degree $\leqslant 2n-1$. This is clear, since for odd p we have

$$\int_{-1}^{1} \sqrt{1-x^2}\, p(x)\, dx = 0, \qquad p(0) \sum_{k \in J} a_k + \sum_{k \notin J} a_k p(t_k) = 0.$$

Therefore the quadrature formula integrates exactly all algebraic polynomials of degree $2n-1$. Then it coincides with the Gaussian quadrature with weight $\sqrt{1-x^2}$ in $[-1, 1]$. Hence $t_1, ..., t_n$ are the zeros of the Tchebycheff polynomial of second kind U_n. In particular, up to rotation, we may assume that I_1 is the line segment with equation $x = t_1$, t_1 being the greatest zero of U_n. Then, according to Theorem 4 from [1], (2.1) coincides with the Gaussian formula from Theorem A and the proof is completed. ∎

We conclude this section with a question concerning Christoffel type extension of the result above.

An integrable non-negative function μ on D is said to be a *weight function* (or briefly a *weight*) on D if it does not vanish on a set of positive measure on D. So far we constructed and proved the uniqueness of the Gaussian formula with a standard constant weight $\mu(x, y) \equiv 1$. One could state the corresponding problem for a general weight μ (The extension is due to Christoffel in the univariate case). The first task is to prove the existence of a quadrature formula of the form

$$\iint_D \mu(x, y)\, f(x, y)\, dx\, dy \approx \sum_{k=1}^{n} A_k \int_{I_k} f(x, y)\, ds \tag{2.4}$$

with ADP as high as possible, namely the existence of a *Gaussian quadrature formula* with weight μ. A weaker version of the existence problem is to look for a formula based on the corresponding weighted line integrals

$$\int_{I_k} \mu(x, y)\, f(x, y)\, ds, \qquad k = 1, ..., n.$$

Even in this form the study of the existence leads to the following interesting open question:

Let μ be a weight function on D. Do the functions

$$x^k \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mu(x, y)\, y^{n-k}\, dy, \qquad k = 0, ..., n,$$

constitute a Tchebycheff system on $[-1, 1]$?

If so, following our method, one can derive from the Krein theorem (see [5]) the existence of a Gaussian quadrature for weighted integrals.

3. GAUSS–LOBATTO FORMULA

We would like to have a multidimensional analogue of the Gauss–Lobatto formula. As such an analogue we consider quadrature of the type

$$\iint_D f(x, y)\,dx\,dy$$

$$\approx B_1 f(x_1, y_1) + B_2 f(x_2, y_2) + \sum_{k=1}^{n} A_k \int_{I_k} f(x, y)\,ds, \tag{3.1}$$

where $\{I_k\}$, $k=1, ..., n$, are n chords in D and $\{(x_i, y_i)\}$, $i=1, 2$, are two points on the unit circle $\{(x, y): x^2+y^2=1\}$. A simple observation is the fact that

$$\text{ADP}(3.1) < 2n+2 \tag{3.2}$$

for each choice of the coefficients A_k, B_i and parameters (t_k, θ_k), (x_i, y_i). To show this, we introduce (in addition to $l_1, ..., l_n$, defined in the previous section) the line l_{n+1} that passes through the points (x_1, y_1) and (x_2, y_2) and consider the corresponding polynomial

$$\omega := l_1 \cdots l_{n+1}$$

associated with formula (3.1). Evidently $\omega^2 \in \pi_{2n+2}(\mathbb{R}^2)$ and $\iint_D \omega^2 > 0$, while

$$B_1 \omega^2(x_1, y_1) + B_2 \omega^2(x_2, y_2) + \sum_{k=1}^{n} A_k \int_{I_k} \omega^2(x, y)\,ds = 0.$$

This proves (3.2).

We call a formula of type (3.1) with $\text{ADP} = 2n+1$ a *Gauss–Lobatto quadrature formula.*

Along with the associated polynomial ω, defined as above, we introduce also the polynomials

$$\omega_k = l_1 \cdots l_{k-1} l_{k+1} \cdots l_{n+1}, \qquad k=1, ..., n+1.$$

The formulae of maximal ADP have the following properties.

LEMMA 3.1. *Let* (3.1) *be a Gauss–Lobatto quadrature formula and let* ω *be its corresponding polynomial. Then the following holds*:

(a) *The polynomial* ω *is orthogonal to every* $Q \in \pi_n(\mathbb{R}^2)$ *on* D *and*

$$\omega(-x, -y) = (-1)^{n+1}\, \omega(x, y). \tag{3.3}$$

(b) *The polynomial* ω_{n+1} *is orthogonal to every* $Q \in \pi_{n-1}(\mathbb{R}^2)$ *on* D *with weight* $(1 - x^2 - y^2)$ *and*

$$\omega_{n+1}(-x, -y) = (-1)^n\, \omega_{n+1}(x, y). \tag{3.4}$$

(c) $(x_1, y_1) = -(x_2, y_2)$ *and* $B_1 = B_2$.

Proof. (a) The polynomial ωQ belongs to $\pi_{2n+1}(\mathbb{R}^2)$ if $Q \in \pi_n(\mathbb{R})^2$. Since (3.1) is a Gauss–Lobatto quadrature, we have

$$\iint_D \omega Q\, dx\, dy = B_1 \omega(x_1, y_1)\, Q(x_1, y_1) + B_2 \omega(x_2, y_2)\, Q(x_2, y_2) + \sum_{k=1}^{n} A_k \int_{I_k} \omega Q = 0.$$

Because of the orthogonality of ω and the fact that D is a central symmetric domain, it follows from a known general result (see [7, p. 164]) that ω has the property (3.3).

(b) Consider $\omega_{n+1} \in \pi_n(\mathbb{R}^2)$ and the weight $(1 - x^2 - y^2)$. Let us apply (3.1) to the polynomial

$$(1 - x^2 - y^2)\, \omega_{n+1} Q \in \pi_{2n+1}(\mathbb{R}^2), \qquad Q \in \pi_{n-1}(\mathbb{R}^2).$$

As a result we get

$$\iint_D (1 - x^2 - y^2)\, \omega_{n+1} Q\, dx\, dy = 0$$

for any polynomial Q of degree $\leqslant n-1$ in $\mathbb{R}^2$. Hence ω_{n+1} is orthogonal to $\pi_{n-1}(\mathbb{R}^2)$ with a central symmetric weight $(1 - x^2 - y^2)$. The domain D is central symmetric too and therefore (see [7]) ω_{n+1} has the property (3.4).

(c) From (3.3), (3.4), and the fact that $\omega = \omega_{n+1} l_{n+1}$ it follows that l_{n+1} passes through the origin. That is, $(x_1, y_1) = -(x_2, y_2)$. Assume now that (x_1, y_1) lies on some of the lines I_k, $k < n+1$. Then, using (3.3) and the property mentioned above, we derive that $\omega_{n+1}(x_1, y_1) = \omega_{n+1}(x_2, y_2) = 0$, which yields that the approximation assigned to $\iint_D \omega_{n+1}^2$ is zero. Thus the formula is not exact for ω_{n+1}^2 from $\pi_{2n}(\mathbb{R}^2)$, a contradiction. Therefore none of these two points belongs to any of the lines I_k, $k < n+1$.

Next, apply (3.1) to the polynomials $(x-x_1)\,\omega_{n+1}^2$ and $(x-x_2)\,\omega_{n+1}^2$ from $\pi_{2n+1}(\mathbb{R}^2)$. We obtain that

$$B_2(x_2-x_1)\,\omega_{n+1}^2(x_2, y_2)=\iint_D (x-x_1)\,\omega_{n+1}^2\,dx\,dy,$$

$$B_1(x_1-x_2)\,\omega_{n+1}^2(x_1, y_1)=\iint_D (x-x_2)\,\omega_{n+1}^2\,dx\,dy.$$

But, as mentioned in (2.3),

$$\iint_D x\omega_{n+1}^2\,dx\,dy=0.$$

In addition, $\omega_{n+1}^2(x_1, y_1)=\omega_{n+1}^2(-x_1, -y_1)=\omega_{n+1}^2(x_2, y_2)$. Since the points (x_i, y_i), $i=1, 2$ are not on any of the lines I_k, $k<n+1$, we have $B_1=B_2$ and the proof is completed. ∎

Now we shall construct a quadrature with a maximal ADP in the set of bivariate polynomials. The construction is based on the classical Gauss–Lobatto formula with weight $\sqrt{1-x^2}$, namely,

$$\int_{-1}^{1}\sqrt{1-x^2}\,p(x)\,dx\approx bp(-1)+bp(1)+\sum_{j=1}^{n} a_j p(x_j), \tag{3.5}$$

which is exact for all polynomials $p\in\pi_{2n+1}(\mathbb{R})$. It is known that $-1, x_1, ..., x_n, 1$ are the zeroes of $U_{n+2}-U_n$. Also, it can be shown that $x_1, ..., x_n$ are the zeroes of U'_{n+1}. Then the following theorem holds.

THEOREM 3.1. *Let $\{a_j\}$ and b be the coefficients of* (3.5) *and let $\{x_j\}$ be the zeros of U'_{n+1}. Then the quadrature formula*

$$\iint_D f(x, y)\,dx\,dy\approx 2bf(-1, 0)+2bf(1, 0)$$
$$+\sum_{j=1}^{n}\frac{a_j}{\sqrt{1-x_j^2}}\int_{-\sqrt{1-x_j^2}}^{\sqrt{1-x_j^2}} f(x_j, y)\,dy, \tag{3.6}$$

is exact for all polynomials $f\in\pi_{2n+1}(\mathbb{R}^2)$.

Proof. Let $f\in\pi_{2n+1}(\mathbb{R}^2)$. Then it can be written in the form

$$f(x, y)=\sum_{k=0}^{2n+1} c_k(x)\,y^k$$

with certain polynomials $c_k \in \pi_{2n-k+1}(\mathbb{R})$. Then

$$\iint_D f(x, y)\, dx\, dy = \sum_{s=0}^{n} \frac{2}{2s+1} \int_{-1}^{1} \sqrt{1-x^2}\, c_{2s}(x)(1-x^2)^s\, dx.$$

Apply now formula (3.5) to $c_{2s}(x)(1-x^2)^s \in \pi_{2n+1}(\mathbb{R})$. We obtain

$$\begin{aligned}
&\iint_D f(x, y)\, dx\, dy \\
&\quad = b \cdot \sum_{s=0}^{n} \frac{2}{2s+1} c_{2s}(-1)(1-(-1)^2)^s \\
&\qquad + b \cdot \sum_{s=0}^{n} \frac{2}{2s+1} c_{2s}(1)(1-(1)^2)^s \\
&\qquad + \sum_{j=1}^{n} a_j \sum_{s=0}^{n} \frac{2}{2s+1} c_{2s}(x_j)(1-x_j^2)^s \\
&\quad = 2bc_0(-1) + 2bc_0(1) + \sum_{j=1}^{n} \frac{a_j}{\sqrt{1-x_j^2}} \int_{-\sqrt{1-x_j^2}}^{\sqrt{1-x_j^2}} f(x_j, y)\, dy \\
&\quad = 2bf(-1, 0) + 2bf(1, 0) + \sum_{j=1}^{n} \frac{a_j}{\sqrt{1-x_j^2}} \int_{-\sqrt{1-x_j^2}}^{\sqrt{1-x_j^2}} f(x_j, y)\, dy,
\end{aligned}$$

and the proof is completed. ∎

Now we are prepared to prove the uniqueness of the constructed formula.

THEOREM 3.2. *There is a unique* (*up to rotation*) *quadrature formula of type* (3.1) *with* ADP $= 2n+1$.

Proof. Let (3.1) be a Gauss–Lobatto quadrature formula. As in the proof of Theorem 2.1, we consider the radial polynomials

$$(x^2+y^2)^m, \qquad m = 0, ..., n,$$

and use (3.1) to calculate the corresponding integral. We obtain that

$$\begin{aligned}
&\int_{-1}^{1} \sqrt{1-x^2}\, p_{2m}(x)\, dx \\
&\qquad = B_1 + B_2 + \sum_{k=1}^{n} A_k \sqrt{1-t_k^2}\, p_{2m}(t_k), \qquad m = 0, ..., n,
\end{aligned}$$

where $p_{2m} \in \pi_{2m}(\mathbb{R})$ is an even algebraic polynomial, defined via the equality

$$\sqrt{1-x^2}\, p_{2m}(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^m \, dy.$$

Since $p_{2m}(-1) = p_{2m}(1) = 2$, the above obtained equalities show that the induced univariate quadrature formula

$$\int_{-1}^{1} \sqrt{1-x^2}\, f(x)\, dx \approx b_1 f(-1) + b_2 f(1) + \sum_{k=1}^{n} a_k f(t_k), \tag{3.7}$$

with $b_i = B_i/2$ and $a_k = A_k \sqrt{1-t_k^2}$, integrates exactly all even polynomials of degree $\leqslant 2n$.

Note that by Lemma 3.1, part (b), the polynomial ω_{n+1} has property (3.4) and therefore its zero lines are pair-wise symmetric, that is,

$$t_k = -t_{n-k+1}, \qquad k = 1, ..., n.$$

As in the proof of Theorem 2.1 we introduce the set $J \subset \{1, ..., n\}$ of indices $\{k\}$ for which $t_k = 0$. If $J \neq \{1, ..., n\}$, we apply (3.1) to ω_k^2, $k \notin J$, $k \neq n+1$, and ω_{n-k+1}^2. Following the proof of Theorem 2.1 and using property (3.3) of ω, we derive that

$$a_k = a_{n-k+1}, \qquad k \notin J.$$

Also, from Lemma 3.1, part (c), we have $b_1 = b_2$. Then (3.7) should be exact for all odd functions, and particularly for all odd polynomials of degree $\leqslant 2n+1$. This is so, because for odd f we have

$$\int_{-1}^{1} \sqrt{1-x^2}\, f(x)\, dx = 0,$$

$$b_1 f(-1) + b_2 f(1) + f(0) \sum_{k \in J} a_k + \sum_{k \notin J} a_k f(t_k) = 0.$$

If $J = \{1, ..., n\}$, then the second sum disappears and the statement still holds. Hence (3.7) coincides with the Gauss–Lobatto quadrature with weight $\sqrt{1-x^2}$ in $[-1, 1]$. Therefore the parameters $t_1, ..., t_n$ of any extremal quadrature formula must coincide with the nodes of the univariate Gauss–Lobatto quadrature formula (3.5).

It remains to show that the chords $I_1, ..., I_n$ are parallel. Let x_1 be the greatest zero of U'_{n+1}. Without loss of generality we can assume that the line I_1 has equation $x = x_1$. From Lemma 3.1, part (b), we know that ω_{n+1} is orthogonal to $\pi_{n-1}(\mathbb{R}^2)$ in D with weight $(1-x^2-y^2)$. Now following the scheme outlined in [12], page 70, we construct the following orthogonal basis for $\pi_n(\mathbb{R}^2)$, corresponding to the weight $(1-x^2-y^2)$,

$$\{(U'_{m+1}(x))^{(j)}\, V_j(x, y)\}_{j=0,\, m=0}^{m,\, n}.$$

Here

$$V_j(x, y)=\begin{cases}(y^2+x^2q_{2s,1}^2-q_{2s,1}^2)\cdots(y^2+x^2q_{2s,s}^2-q_{2s,s}^2), \\ \qquad \text{if} \quad j=2s, \\ y(y^2+x^2q_{2s+1,1}^2-q_{2s+1,1}^2)\cdots(y^2+x^2q_{2s+1,s}^2-q_{2s+1,s}^2), \\ \qquad \text{if} \quad j=2s+1,\end{cases}$$

with $q_{j,l}$, $l=1, ..., [j/2]$, being the positive zeros of the jth orthogonal polynomial on $[-1, 1]$ with weight $(1-x^2)$. Then we have

$$\omega_{n+1}(x, y)=\sum_{k=1}^{n+1} c_k\, U_{n+1}^{(k)}(x)\, V_{k-1}(x, y)$$

with certain constant coefficients $\{c_k\}$. But $\omega_{n+1}(x_1, y)=0$ for every y. Then, taking into account that $\{V_{k-1}(x_1, y)\}_1^{n+1}$ are linearly independent polynomials, after comparison of the coefficients we conclude that

$$c_k\, U_{n+1}^{(k)}(x_1)=0, \qquad k=1, ..., n+1.$$

Since x_1 is the greatest zero of U'_{n+1}, all zeroes of $U_{n+1}^{(k)}$, $k>1$, belong to the interval $(-1, x_1)$. Then $c_k=0$, $k=2, ..., n+1$, and therefore

$$\omega_{n+1}(x, y)=c_1\, U'_{n+1}(x), \qquad c_1\neq 0.$$

From here and part (c) of Lemma 3.1 it follows that (3.1) has the form

$$\iint_D f(x, y)\, dx\, dy \approx Bf(x_1, y_1)+Bf(-x_1, -y_1)$$
$$+\sum_{j=1}^{n} A_j \int_{-\sqrt{1-x_j^2}}^{\sqrt{1-x_j^2}} f(x_j, y)\, dy.$$

Note that the line l_{n+1} passes through the origin. Then ω can be written as

$$\omega(x, y)=(\beta x+\gamma y)\, U'_{n+1}(x)$$

with some real coefficients β and γ. We use the fact that (see [1])

$$\{U_m^{(j)}(x)\, W_j(x, y)\}_{j=0,\, m=0}^{m,\, n+1}$$

is an orthogonal basis for $\pi_{n+1}(\mathbb{R}^2)$ with weight $\mu(x)=1$. Here

$$W_j(x, y)=\begin{cases}(y^2+x^2p_{2s,1}^2-p_{2s,1}^2)\cdots(y^2+x^2p_{2s,s}^2-p_{2s,s}^2), \\ \qquad \text{if} \quad j=2s, \\ y(y^2+x^2p_{2s+1,1}^2-p_{2s+1,1}^2)\cdots(y^2+x^2p_{2s+1,s}^2-p_{2s+1,s}^2), \\ \qquad \text{if} \quad j=2s+1,\end{cases}$$

with $p_{j,l}$, $l=1, ..., [j/2]$, the positive zeros of the jth Legendre polynomial. By part (a) of Lemma 3.1, ω is orthogonal to $\pi_n(\mathbb{R}^2)$ in D. Therefore we have the representation

$$(\beta x+\gamma y)\, U'_{n+1}(x)=\sum_{k=0}^{n+1} c_k\, U_{n+1}^{(k)}(x)\, W_k(x, y).$$

After we compare the coefficients in front of the powers of y, we get that

$$c_2=\cdots=c_{n+1}=0, \qquad \gamma U'_{n+1}(x)=c_1\, U'_{n+1}(x),$$
$$\beta x U'_{n+1}(x)=c_0\, U_{n+1}(x).$$

It follows from the last equation that $\beta=c_0=0$. Hence $\omega(x, y)=\gamma y U'_{n+1}(x)$ and this gives $(x_1, y_1)=(-1, 0)$. Therefore all nodes and chords of (3.1) coincide with those of formula (3.5) given in Theorem 3.1. Then the equalities

$$B=2b, \qquad A_k=\frac{a_k}{\sqrt{1-x_k^2}},$$

follow and the proof is completed. ▮

Remark 3.1. We proved (see Theorem A) that (1.3) is exact for all polynomials from $\pi_{2n-1}(\mathbb{R}^2)$. Let us go further and represent the line integrals over I_k by the corresponding one dimensional Gaussian formulae

$$\int_{-\sin(k\pi/(n+1))}^{\sin(k\pi/(n+1))} f\left(\cos\frac{k\pi}{n+1}, y\right) dy \approx \sum_{j=1}^{n} B_{j,k} f\left(\cos\frac{k\pi}{n+1}, y_{j,k}\right)$$

that are exact for all polynomials in $\pi_{2n-1}(\mathbb{R})$. Then we arrive at the quadrature

$$\iint_D f(x, y)\, dx\, dy$$
$$\approx\frac{\pi}{n+1}\sum_{k=1}^{n}\sin\frac{k\pi}{n+1}\sum_{j=1}^{n} B_{j,k} f\left(\cos\frac{k\pi}{n+1}, y_{j,k}\right) \tag{3.8}$$

which uses n^2 point evaluations and has ADP $\geqslant 2n-1$. There was a considerable effort over the years to find formulae with minimal number $N^*(n)$ of nodes that integrate exactly all polynomials from $\pi_{2n-1}(\mathbb{R}^2)$. These formulae are called *minimal*. Particular minimal ones have been constructed for polynomials of a certain small degree (see, for example, [7]). It is shown in [6], that for centrally symmetric weight functions and domains, $N^*(n) \geqslant n(n+1)/2+[n/2]$. On the other hand, one can always find an interpolation set of $n(2n+1)$ nodes and construct the interpolatory type quadrature formula which will have an ADP $=2n^2+n=: N_0(n)$. Thus any formula that uses less than $N_0(n)$ nodes is of interest. Note that (3.8) has roughly twice as many nodes as the corresponding minimal quadrature and twice less than the interpolatory one, which is a quite good property.

One may proceed the same way starting from the Gauss–Lobatto formula (3.6), and derive a quadrature with algebraic degree of precision at least $2n-1$ that uses n^2-2n+3 nodes (which is a smaller number than in the previous case). Moreover, the coefficients and the nodes can be given explicitly.

4. MULTIVARIATE EXTENSION

For every $\mathbf{x}=(x_1, ..., x_d)$ from $\mathbb{R}^d$ we set

$$\|\mathbf{x}\| := (x_1^2 + \cdots + x_d^2)^{1/2},$$

$$\mathbf{B}^d := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leqslant 1\}.$$

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$, that is,

$$S^{d-1} = \partial \mathbf{B} := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}.$$

With every vector $\xi \in S^{d-1}$, $\xi=(\xi_1, ..., \xi_d)$, $\xi_d \geqslant 0$, and a number t we associate the hyperplane $\beta(\xi, t)$ that is perpendicular to ξ and passes through the point $t\xi$. We consider also the corresponding linear polynomial

$$\beta(\xi, t)(\mathbf{x}) := \mathbf{x} \cdot \xi - t, \qquad (\mathbf{x} \cdot \xi := x_1\xi_1 + \cdots + x_d\xi_d).$$

Recall that the *Radon projection* of a scalar valued function f on $\mathbf{B}^d$ is given by

$$\mathcal{R}(f; \xi, t) := \int_{\beta(\xi, t) \cap \mathbf{B}^d} f, \qquad -1 \leqslant t \leqslant 1.$$

Usually $\mathcal{R}(f; \xi, t)$ is considered as a parameterized by ξ family of univariate functions and is called *Radon transform* of f. As is known, the function f can be reconstructed from its Radon transform (see [9] for further results and details).

In this section we consider the problem of recovery of the weighted integral of an algebraic polynomial f on $\mathbf{B}^d$, using minimal number of its Radon projections. In other words, for a given weight $\mu(\mathbf{x})$ on $\mathbf{B}^d$ we study quadrature formulae of the form

$$\int_{\mathbf{B}^d} \mu(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x} \approx \sum_{k=1}^{n} A_k \mathscr{R}(f;\, \xi_k,\, t_k) \tag{4.1}$$

of highest ADP. We allow some of the t_k's to be equal to 1. Then we interpret $\mathscr{R}(f;\, \xi_k,\, t_k)$ as $f(\xi_k)$. Such an interpretation is justified by the continuity argument

$$\lim_{t_k \to 1} \frac{1}{\mathrm{Vol}_{d-1}\{\beta(\xi_k,\, t_k) \cap \mathbf{B}^d\}} \int_{\beta(\xi_k,\, t_k)} f = f(\xi_k).$$

It can be easily derived that

$$\mathrm{ADP}(4.1) < n + n_0,$$

where n_0 is the number of t_k's for which $|t_k| < 1$. Indeed, let us apply (4.1) to the polynomial

$$h(\mathbf{x}) := \prod_{|t_k| < 1} \beta^2(\xi_k,\, t_k)(\mathbf{x}) \prod_{|t_k| = 1} \beta(\xi_k,\, t_k)(\mathbf{x}).$$

Note that for $|t_k| = 1$ the hyperplane $\beta(\xi_k,\, t_k)$ is tangent to $\mathbf{B}^d$ at ξ and hence $\beta(\xi_k,\, t_k)(\mathbf{x})$ does not change sign for $\mathbf{x} \in \mathbf{B}^d$. Then the weighted integral of h on $\mathbf{B}^d$ is non-zero, while the approximate value given by (4.1) is zero. Since h is of degree $n + n_0$, our claim is proved.

Further we use projections determined by the vector $\xi = (1, 0, ..., 0)$. In this case we omit ξ and write simply

$$\mathscr{R}(f;\, t) = \int_{B(t)} f, \qquad -1 \leqslant t \leqslant 1,$$

where $B(t)$ is the intersection of $\mathbf{B}^d$ and the hyperplane in $\mathbb{R}^d$ which is perpendicular to the Ox_1 axis and passes through $(t, 0, ..., 0)$. The next auxiliary lemma shows that the Radon projection $\mathscr{R}(f;\, \cdot\,)$ of a polynomial f from $\pi_n(\mathbb{R}^d)$ is a weighted univariate polynomial.

Lemma 4.1. *For each $f \in \pi_n(\mathbb{R}^d)$ there is a polynomial p from $\pi_n(\mathbb{R})$ such that*

$$\mathscr{R}(f;\, t) = (1 - t^2)^{(d-1)/2}\, p(t). \tag{4.2}$$

Furthermore,

$$p(-1) = \text{Vol}\,\mathbf{B}^{d-1} f(-1, 0, ..., 0),$$
$$p(1) = \text{Vol}\,\mathbf{B}^{d-1} f(1, 0, ..., 0). \tag{4.3}$$

Proof. Notice that for $d=2$ this lemma was proved already in [1] and in the previous sections.

It is sufficient to show (4.2) for monomials

$$\mathbf{x}^{\bar{m}} = x_1^{m_1} \cdots x_d^{m_d}, \qquad \bar{m} := (m_1, ..., m_d).$$

Let $\bar{m}(2, d) := (m_2, ..., m_d)$. We have

$$\mathcal{R}(\mathbf{x}^{\bar{m}}; t) = \int_{B(t)} t^{m_1} x_2^{m_2} \cdots x_d^{m_d}\, dx_2 \cdots dx_d$$
$$= t^{m_1} \int_{\{\mathbf{y} \in \mathbb{R}^{d-1}:\ \|\mathbf{y}\| \leqslant r\}} \mathbf{y}^{\bar{m}(2, d)}\, d\mathbf{y}$$

with $r := \sqrt{1-t^2}$. Clearly the last integral is zero if at least one m_j, $j = 2, ..., d$, is odd and then $p(t) \equiv 0$ is the wanted polynomial. Let us assume now that all m_j, $j = 2, ..., d$, are even. After the change of variables $y_k = r z_k$, $k = 1, ..., d-1$, we get

$$\mathcal{R}(\mathbf{x}^{\bar{m}}; t) = t^{m_1} r^{|\bar{m}(2, d)|} r^{d-1} \int_{B^{d-1}} \mathbf{z}^{\bar{m}(2, d)}\, d\mathbf{z}$$
$$= C(1-t^2)^{(d-1)/2}\, t^{m_1} (1-t^2)^{|\bar{m}(2, d)|/2},$$

where $C = \int_{B^{d-1}} \mathbf{z}^{\bar{m}(2, d)}\, d\mathbf{z}$ is a constant and $|\bar{m}(2, d)| := m_2 + \cdots + m_d$. The relation is proved. To verify (4.3) we just observe that

$$\mathcal{R}(f; t) = \int_{B(t)} f = r^{d-1} \int_{\mathbf{B}^{d-1}} f(t, rz_1, ..., rz_{d-1})\, d\mathbf{z} = r^{d-1} p(t).$$

Thus

$$p(t) = \frac{\text{Vol}\,\mathbf{B}^{d-1}}{\text{Vol}_{d-1} B(t)} \int_{B(t)} f, \tag{4.4}$$

and clearly

$$p(1) = \lim_{t \to 1} p(t) = \text{Vol}\,\mathbf{B}^{d-1} \lim_{t \to 1} \frac{1}{\text{Vol}_{d-1} B(t)} \int_{B(t)} f(\mathbf{x})\, d\mathbf{x}$$
$$= \text{Vol}\,\mathbf{B}^{d-1} f(1, 0, ..., 0). \quad \blacksquare$$

We are going to prove the existence of Gaussian quadrature formulae on $\mathbf{B}^d$ in the case of a ridge weight function. Recall that a function G on $\mathbb{R}^d$ is called a *ridge function* if $G(\mathbf{x}) = g(\xi \cdot \mathbf{x})$ for some $\xi \in S^{d-1}$ and a univariate function g.

Our proof is constructive and relies on the following theorem which reveals a one-to-one correspondence between the univariate quadratures on $[-1, 1]$ and a class of quadrature formulae on $\mathbf{B}^d$.

THEOREM 4.1. *Assume that μ is an arbitrary weight function on $[-1, 1]$ and d is a natural number. The quadrature formula*

$$\int_{-1}^{1} \mu(t)(1-t^2)^{(d-1)/2}\, p(t)\, dt \approx \sum_{k=1}^{n} a_k p(\zeta_k), \tag{4.5}$$

where $-1 \leqslant \zeta_1 < \cdots < \zeta_n \leqslant 1$ integrates exactly all polynomials from $\pi_N(\mathbb{R})$ if and only if the formula

$$\int_{\mathbf{B}^d} \mu(x_1)\, f(\mathbf{x})\, d\mathbf{x} \approx \sum_{k=1}^{n} \frac{a_k}{(1-\zeta_k^2)^{(d-1)/2}} \int_{B(\zeta_k)} f(\mathbf{x})\, d\mathbf{x} \tag{4.6}$$

is exact for all elements in $\pi_N(\mathbb{R}^d)$.

If $\zeta_1 = -1$ (or $\zeta_n = 1$), the corresponding term in (4.6) *has to be interpreted as a_1* $\operatorname{Vol} \mathbf{B}^{d-1} f(-1, 0, ..., 0)$ (*or a_n* $\operatorname{Vol} \mathbf{B}^{d-1} f(1, 0, ..., 0)$).

Proof. Note that (4.6) may be written in the following equivalent form

$$\int_{\mathbf{B}^d} \mu(x_1)\, f(\mathbf{x})\, d\mathbf{x} \approx \sum_{k=1}^{n} a_k \frac{\operatorname{Vol} \mathbf{B}^{d-1}}{\operatorname{Vol}_{d-1} B(\zeta_k)} \int_{B(\zeta_k)} f(\mathbf{x})\, d\mathbf{x},$$

which covers the cases when some of the ζ_k's are endpoints of the interval $[-1, 1]$. Now the proof of the theorem is immediate. Let $f \in \pi_N(\mathbb{R}^d)$ and (4.5) be exact for all polynomials in $\pi_N(\mathbb{R})$. By Lemma 4.1 we have

$$\int_{\mathbf{B}^d} \mu(x_1)\, f(\mathbf{x})\, d\mathbf{x} = \int_{-1}^{1} \mu(t)\, \mathscr{R}(f; t)\, dt = \int_{-1}^{1} \mu(t)(1-t^2)^{(d-1)/2}\, p(t)\, dt$$

$$= \sum_{k=1}^{n} a_k p(\zeta_k)$$

and by (4.4) we arrive at the desired equality.

Conversely, if (4.6) is exact for all $f \in \pi_N(\mathbb{R}^d)$, then it integrates exactly the ridge monomials x_1^l, $l=0, ..., N$. This leads to the fact that (4.5) is exact for every polynomial $p(t)=t^l$, $l=0, ..., N$, and thus for every $p \in \pi_N(\mathbb{R})$. ∎

Theorem 4.1 shows how a univariate quadrature formula generates its multivariate counterpart. In particular, if (4.5) is Gaussian, then (4.6) is also Gaussian. Thus the existence problem for a quadrature formulae of type (4.1) of maximal ADP is settled in the case of a ridge weight μ.

Next we prove the uniqueness in the case $\mu(\mathbf{x})=1$. We shall follow the idea already demonstrated in Sections 2 and 3. We have used there a certain representation (given in [12], p. 70) of orthogonal polynomials of two variables. The next proposition, which provides a similar representation in the general d-variate case, $d \geqslant 2$, was kindly communicated to us by Yuan Xu. We present below his elegant proof in a separate lemma.

Consider the weight $\mu_\lambda(\mathbf{x}) := (1-\|\mathbf{x}\|^2)^{\lambda-1/2}$ on $\mathbf{B}^d$. Let $V_n^d(\mu_\lambda)$ denote the space of orthogonal polynomials of degree n with respect to μ_λ on $\mathbf{B}^d$. For each $k=0, ..., n$, let $\{Q_\alpha^k\}_{|\alpha|=k}^n$, $\alpha \in N_0^{d-1} := \{\alpha=(\alpha_1, ..., \alpha_{d-1}), \alpha_i \geqslant 0, \alpha_i \in N\}$, be an orthonormal basis for $V_n^{d-1}(\mu_\lambda)$. For $\mathbf{x}=(x_1, \mathbf{y}) \in \mathbf{B}^d$, define

$$P_{\alpha,k}^n(\mathbf{x}) := C_{n-k}^{(k+\lambda+(d-1)/2)}(x_1)(1-x_1^2)^{k/2}\, Q_\alpha^k\left(\frac{\mathbf{y}}{\sqrt{1-x_1^2}}\right),$$

$$|\alpha|=k, \quad \alpha \in N_0^{d-1},$$

where $C_n^{(\beta)}$ denotes the orthogonal Gegenbauer polynomial of degree n, that is,

$$\int_{-1}^{1} C_n^{(\beta)}(t)\, C_m^{(\beta)}(t)(1-t^2)^{\beta-1/2}\, dt = \delta_{nm},$$

with δ_{nm} being the Kronecker symbol.

LEMMA 4.2. *The set* $\{P_{\alpha,k}^n(\mathbf{x})\colon |\alpha|=k, \alpha \in N_0^{d-1}, 0 \leqslant k \leqslant n\}$ *forms an orthonormal basis of* $V_n^d(\mu_\lambda)$

Proof. After a change of variables we get

$$\int_{\mathbf{B}^d} f(\mathbf{x})\, \mu_\lambda(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{-1}^{1} \int_{\mathbf{B}^{d-1}} f(x_1, \sqrt{1-x_1^2}\, \mathbf{y})(1-\|\mathbf{y}\|^2)^{\lambda-1/2}\, d\mathbf{y} (1-x_1^2)^{\lambda+(d-2)/2}\, dx_1.$$

Thus, we have

$$\int_{\mathbf{B}^d} P^n_{\alpha,k}(\mathbf{x})\, P^m_{\beta,j}(\mathbf{x})\, \mu_\lambda(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{-1}^{1} C^{k+\lambda+(d-1)/2}_{n-k}(x_1)$$

$$\times C^{j+\lambda+(d-1)/2}_{m-j}(x_1)(1-x_1^2)^{(k+j)/2+\lambda+(d-2)/2}\, dx_1$$

$$\times \int_{\mathbf{B}^{d-1}} Q^k_\alpha(\mathbf{y})\, Q^j_\beta(\mathbf{y})(1-\|\mathbf{y}\|^2)^{\lambda-1/2}\, d\mathbf{y}$$

$$= \delta_{n,m}\delta_{k,j}\delta_{\alpha,\beta}.$$

This proves the orthogonality. Moreover, we have

$$\sum_{k=0}^{n} \#\{\alpha \in N_0^{d-1} : |\alpha| = k\} = \sum_{k=0}^{n} \binom{k+d-2}{k} = \binom{n+d-1}{n}$$

$$= \dim V_n^d(\mu_\lambda)$$

and thus the polynomials constitute a basis in $V_n^d(\mu_\lambda)$. The proof is completed. ▮

Now we are ready to prove the uniqueness.

THEOREM 4.2. *For every natural n and dimension d there is a unique* (*up to rotation*) *quadrature formula of the form* (4.1) *with* $\mu(\mathbf{x}) \equiv 1$ *which integrates exactly all algebraic polynomials from* $\pi_{2n-1}(\mathbb{R}^d)$.

Proof. Assume that (4.1) with $\mu(\mathbf{x}) \equiv 1$ is a quadrature formula of ADP $= 2n-1$. Let $\omega(\mathbf{x}) := \beta(\xi_1, t_1)(\mathbf{x}) \cdots \beta(\xi_n, t_n)(\mathbf{x})$ be the associated with this formula polynomial. Then, clearly,

$$\int_{\mathbf{B}^d} \omega q = \sum_{k=1}^{n} A_k \int_{\beta(\xi_k, t_k) \cap \mathbf{B}^d} \omega q = 0$$

for every polynomial q from $\pi_{n-1}(\mathbb{R}^d)$. Thus

$$\omega(\mathbf{x}) \text{ is orthogonal to } \pi_{n-1}(\mathbb{R}^d).$$

Then, since $\mathbf{B}^d$ is a symmetric domain and $\mu(\mathbf{x}) \equiv 1$ is a symmetric weight, a general result from the theory of orthogonal polynomials (see, for example, [12]) implies that ω is centrally symmetric. Therefore, if $\beta(\xi_k, t_k)(\mathbf{x})$ is a factor of ω, then $\beta(\xi_k, -t_k)(\mathbf{x})$ is also a factor. As a consequence, we get as in Section 2 that

$$t_k = -t_{n-k+1}, \qquad A_k = A_{n-k+1}, \tag{4.7}$$

for all $k = 1, ..., n$ for which $t_k \neq 0$. Next the proof repeats the arguments presented in Section 2: If (4.1) integrates exactly all polynomials of degree $2n-1$, then it is exact, in particular, for the even symmetric polynomials

$$(x_1^2 + \cdots + x_d^2)^m, \qquad m = 0, ..., n-1.$$

This leads to the fact that the associated univariate quadrature formula (with $a_k/(1-\zeta_k^2)^{(d-1)/2} = A_k$)

$$\int_{-1}^{1} (1-t^2)^{(d-1)/2}\, p(t)\, dt \approx \sum_{k=1}^{n} a_k\, p(t_k) \tag{4.8}$$

is exact for all even polynomials $p(t) = t^{2m}$, $m = 0, ..., n-1$. In addition, (4.7) implies that the quadrature (4.8) is exact also for all odd polynomials (since the nodes $\{t_k\}$ and the coefficients $\{a_k\}$ are symmetric). Therefore (4.8) has ADP equal to $2n-1$. Then it coincides with the Gauss quadrature formula corresponding to the weight $(1-t^2)^{(d-1)/2}$ and thus, the parameters $\{t_k\}$ are determined uniquely. They are the zeros of the orthogonal Gegenbauer polynomial $C_n^{(d/2)}$.

It remains only to show that the directions $\{\xi_k\}_{k=1}^n$ are equal. To do this we proceed as in the proof of Theorem 4 from [1].

Assume that P is an orthogonal polynomial of degree n and

$$P(\mathbf{x}) = (x_1 - a)\, P_1(\mathbf{y}), \qquad \mathbf{x} = (x_1, \mathbf{y}) \in \mathbb{R}^d,$$

where a is the largest zero of $C_n^{(\lambda + (d-1)/2)}$. We shall show that

$$P(\mathbf{x}) = \text{const} \cdot C_n^{(\lambda + (d-1)/2)}(x_1).$$

Indeed, since $P \in V_n^d(\mu_\lambda)$, we can write

$$P(\mathbf{x}) = \sum_{k=0}^{n} \sum_{|\alpha| = k} a_{\alpha, k}\, C_{n-k}^{(k+\lambda+(d-1)/2)}(x_1)(1-x_1^2)^{k/2}\, Q_\alpha^k\left(\frac{\mathbf{y}}{\sqrt{1-x_1^2}}\right).$$

The polynomial P vanishes on the line $x_1 = a$. Therefore

$$0 = P(a, \mathbf{y}) = \sum_{k=0}^{n} \sum_{|\alpha| = k} a_{\alpha, k}\, C_{n-k}^{(k+\lambda+(d-1)/2)}(a)(1-a^2)^{k/2}\, Q_\alpha^k\left(\frac{\mathbf{y}}{\sqrt{1-a^2}}\right).$$

Since $\{Q_\alpha^k\}_{k=0,\,|\alpha|=k}^{n}$ are linearly independent and, in fact (by Lemma 4.2), form a basis for $\pi_n(\mathbb{R}^{d-1})$, we conclude that

$$a_{\alpha,k} C_{n-k}^{(k+\lambda+(d-1)/2)}(a)(1-a^2)^{k/2}=0, \qquad |\alpha|=k, \quad k=0, ..., n.$$

Since

$$C_{n-k}^{(k+\lambda+(d-1)/2)}(t)=\text{const}\cdot\left(\frac{d}{dt}\right)^k C_n^{(\lambda+(d-1)/2)}(t),$$

it follows from the interlacing property of the zeros of the Gegenbauer polynomials that

$$C_{n-k}^{(k+\lambda+(d-1)/2)}(a)\neq 0, \qquad 1\leqslant k\leqslant n.$$

Hence $a_{\alpha,k}=0$, for $|\alpha|=k$, $k=1, ..., n$. Therefore

$$P(\mathbf{x})=a_{0,0}C_n^{(\lambda+(d-1)/2)}(x_1).$$

This completes the proof of our claim. It yields that the directions $\{\xi_k\}$ of any orthogonal polynomial ω, with parameters $\{t_k\}$ coinciding with the zeros of the Gegenbauer polynomials, should be equal. Thus ω is uniquely determined up to rotation. The uniqueness of the Gaussian quadrature formula (4.1) with a weight function $\mu(\mathbf{x})\equiv 1$ is proved. ∎

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